10K MATHS EXERCISES & PROBLEMS

Standards - Based Illustrated Narratives

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Alecos Papadatos

Keanu (key-ah-noo) is looking at numbers arranged in order. This sequence has no end; it's infinite. The rule that links the numbers one after the other is called a rate and is less than one. That means each number is smaller than the one before it. Finally, the numbers will get incredibly small, almost zero, as the sequence lists more of them. But none will ever be zero no matter how long the list is. And to answer Keanu's question, yes we know this without ever actually seeing the sequence.
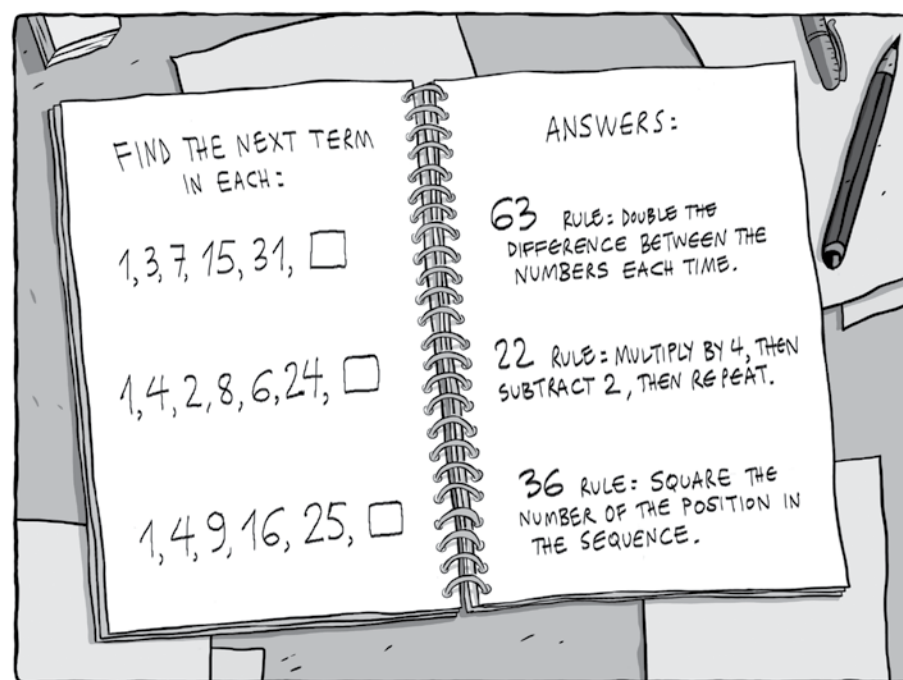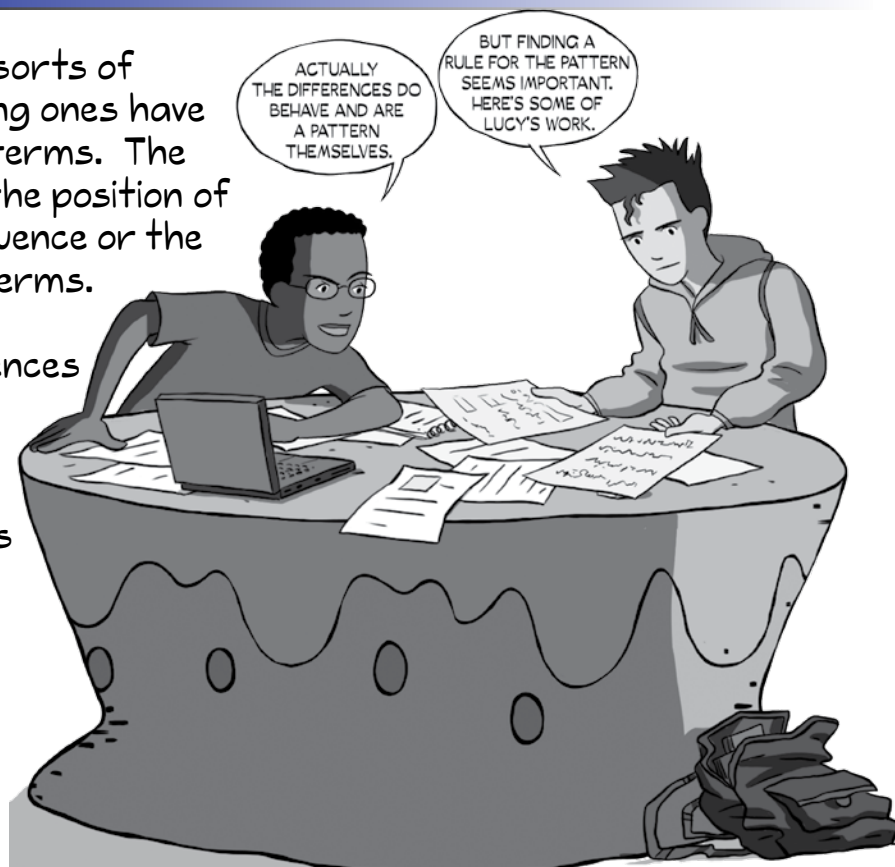
Lucy and Jordan are Keanu's passengers here. Let's join them to find out more.

Sequences come in all sorts of flavors. The interesting ones have patterns to find new terms. The secret is to consider the position of the number in the sequence or the differences between terms. Be careful because sometimes the differences are always the same between terms of one sequence, and for others the differences change.

ACTUALLY THE DIFFERENCES DO BEHAVE AND ARE A PATTERN THEMSELVES.

BUT FINDING A RULE FOR THE PATTERN SEEMS IMPORTANT. HERE'S SOME OF LUCY'S WORK.

A rule is used to find the value of new or missing terms in a sequence. Lucy has written rules for these sequences, but these may not be the only ones for them. She based the first two on the numbers that she sees. But in the last one, her rule depends on the position of the number in the sequence.

FIND THE NEXT TERM IN EACH:

1, 3, 7, 15, 31, ☐

1, 4, 2, 8, 6, 24, ☐

1, 4, 9, 16, 25, ☐

ANSWERS:

63   RULE: DOUBLE THE DIFFERENCE BETWEEN THE NUMBERS EACH TIME.

22   RULE: MULTIPLY BY 4, THEN SUBTRACT 2, THEN REPEAT.

36   RULE: SQUARE THE NUMBER OF THE POSITION IN THE SEQUENCE.

Some sequences are classics. That is you see them, or similar ones, time and again. Here's our list.

CLASSIC: ARITHMETIC (say air ith met ik) The difference between one term and the next is always the same.
An example is 3, 6, 9, 12 ...
(... means that the pattern continues without end)

CLASSIC: GEOMETRIC Each term after the first is the previous term times a nonzero fixed number.
An example is 1,3, 9, 27 ...

CLASSIC: TRIANGULAR Starting with 1 and 3, the difference between two terms is always one more than the difference between the previous two.
This sequence is 1, 3, 6, 10, ...

THESE CLASSICS ARE A GOOD START, BUT THE MOST INTERESTING WORK IS SOLVING OTHERS. DON'T YOU AGREE?

Mmm.

CLASSIC: SQUARES Each term is the square of its position in the sequence.
This sequence is 1, 4, 9, 16, ...

CLASSIC: FIBONACCI Each term after beginning with 0 and 1 (or 1 and 1) is the sum of the previous two terms.
An example is 0,1,1,2,3, ...
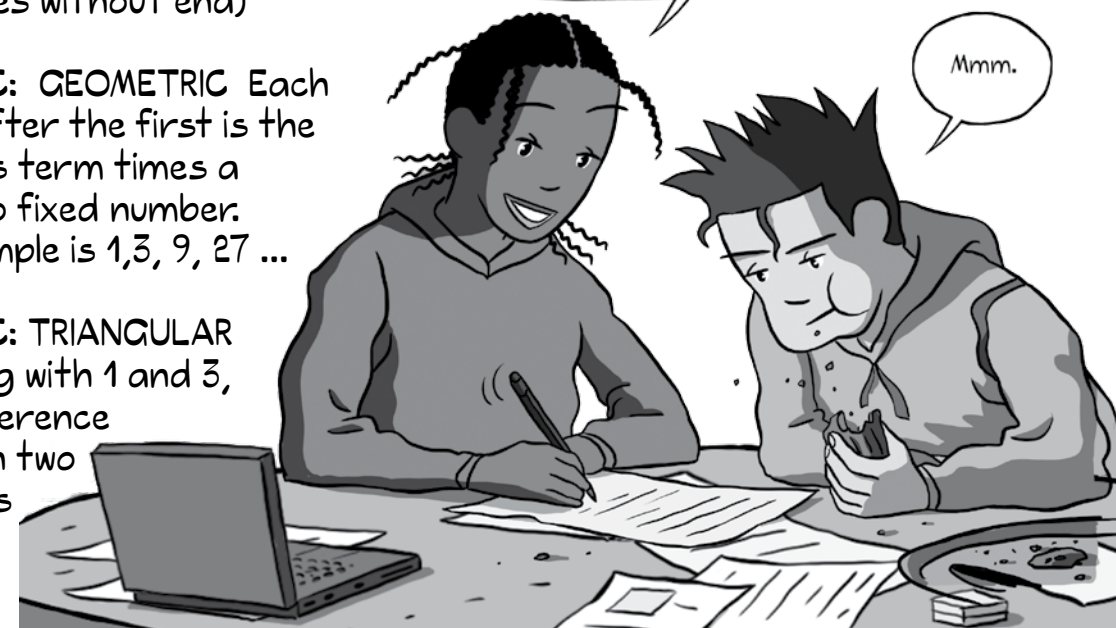
Can you match these patterns with the sequences below?

1._____ Each term after the first one is twice the previous term 2, 4, 12, 48, ...

2._____ Each term after the first two is the sum of the previous two terms.

3._____ Each term of the sequence 1, 3, 5 ... is increased by 6.

4._____ Each term after the first one is one-half the previous term.

5._____ Each term is a square of the terms in 3, 5, 7, ...

6._____ Each term after the first two terms, 4 and 6, increases the difference between terms by 1.

7._____ Each term is the sum of matching terms of the sequences 1, 3, 5, ... and 3, 5, 7, ...

8._____ Each term after the first one is twice the previous term plus 1.

9.   Choose one of the sequences that you did not use as an answer and describe it here.

a .4, 8, 12, 16, ...	e. 4, 6, 10, 18, ...	i. 7, 9, 11, ...
b. 9, 25, 49, ...	f. 3, 6, 12, ...	j. 1, 3, 7, 15, ...
c. 36, 18, 9, ...	g. 1, 1, 1, 1, ...	k. 36, 24, 16, ...
d. 4, 6, 9, 13,	h. 1, 2, 3, 5, ...	

Exercises

Please find the next term and the rule for each of the sequences below.

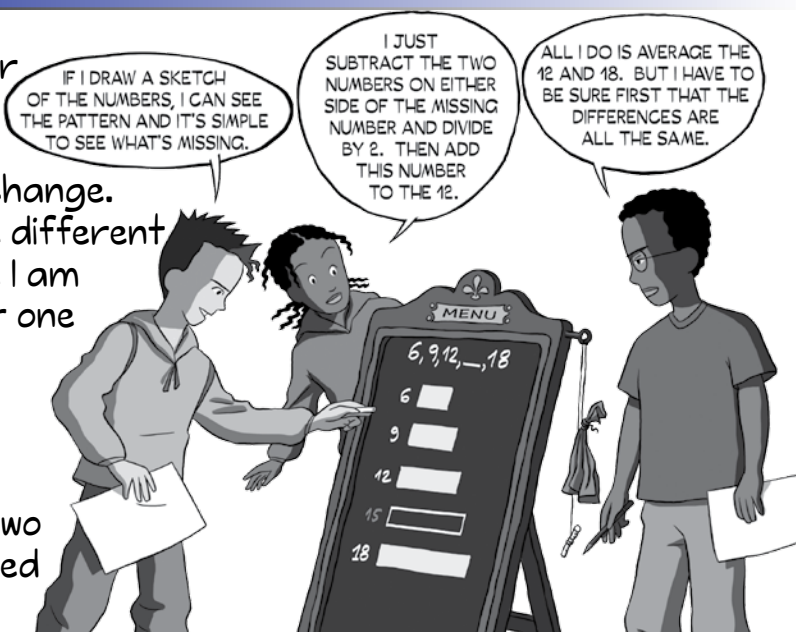| Sequence	Rule
3, 6, 12, 24, _____	
16, 15, 13, 10, 6, _____	
60, 56, 48, 32, _____	
2, 8, 18, 32, _____	
58, 47, 36, 25, _____	
1, 2, 3, 5, _____	
48, 24, 12, 6, _____	
5, 10, 8, 16, _____	
4, 13, 40, 121, _____	
1, 5, 13, 25, _____	

Finding the missing number in a sequence can be solved if the difference between terms does not change. These kids show you three different ways, and there are more I am sure. Use one of these, or one you make up, to solve the sequences below.

Also, Keanu wanted you to know that an average of two numbers is their sum divided by 2.

IF I DRAW A SKETCH OF THE NUMBERS, I CAN SEE THE PATTERN AND IT'S SIMPLE TO SEE WHAT'S MISSING.

I JUST SUBTRACT THE TWO NUMBERS ON EITHER SIDE OF THE MISSING NUMBER AND DIVIDE BY 2. THEN ADD THIS NUMBER TO THE 12.

ALL I DO IS AVERAGE THE 12 AND 18. BUT I HAVE TO BE SURE FIRST THAT THE DIFFERENCES ARE ALL THE SAME.

MENU
6, 9, 12, ___, 18
6
9
12
15
18

Exercises.

Please find the missing term in each sequence.

Please write its rule.

Sequence	Rule
17, 14, _____, 8, 5	
26, 31, _____, 41, 46	
73, 64, 55, _____, 37	
8, 9.5, _____, 12.5, 14	
100, 78, 56, _____, 12	
11, 22.5, _____, 45.5, 57	

Jordan is correct to a certain extent. Missing two numbers in a row from a brief sequence could cause a real challenge. Luckily this sequence and the one below all have the same difference between terms.

Lucy has been able to expand her previous routine to find two missing terms instead of just one. But Keanu has to think quite a bit whether his way will work with two missing numbers. He wants you to know that he could use your help in the box below.

HEY, TWO MISSING NUMBERS! NOW WE HAVE A DECENT PROBLEM.

I'M GOOD. I JUST DIVIDE THE DIFFERENCE BETWEEN 12 AND 18 BY 3 AND ADD IT TWICE.

IF THERE WERE THREE MISSING NUMBERS, I COULD AVERAGE AND AVERAGE AGAIN. BUT WITH THIS, I WILL NEED TO THINK THIS THROUGH.

MENU
12, ___, ___, 18
12
14
16
18

Exercises.

Please find the missing term in each sequence.

Please write its rule.

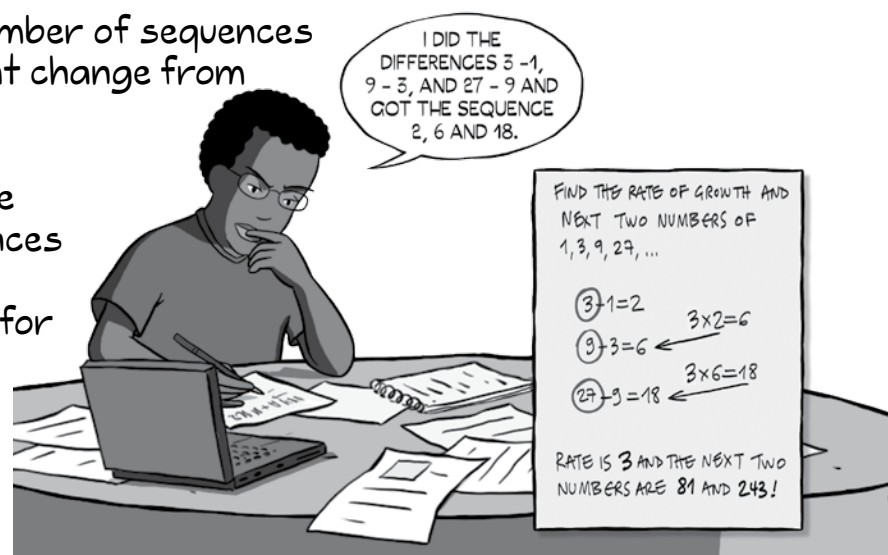| Sequence	Rule
8, 6, _____, _____, 0	
7, _____, _____, 27, 34	
14, _____, _____, 6.5, 4	
18, 30, _____, _____, 66	
30, 24.5, _____, _____, 8	
0, _____, _____, 18, 24	

Of course, a vast number of sequences have differences that change from term to term.

Keanu considered the patterns of differences when he solved the sequence, 1, 3, 9, 17, for the next two terms. Because of this, Keanu knew that he had a classical geometric sequence with a rate of 3.

Keanu did want me to tell you that he had worked on this one before, but wanted you to see his pencil and paper calculations.

I DID THE DIFFERENCES 3 -1, 9 - 3, AND 27 - 9 AND GOT THE SEQUENCE 2, 6 AND 18.

FIND THE RATE OF GROWTH AND NEXT TWO NUMBERS OF 1, 3, 9, 27, ...

③ -1 = 2
⑨ -3 = 6   3 × 2 = 6
㉗ -9 = 18   3 × 6 = 18

RATE IS **3** AND THE NEXT TWO NUMBERS ARE **81** AND **243**!

Exercises.

Please solve the sequences below by considering the pattern of differences.

Sequence	Rule
10, 8, 16, 14, 22, _____, _____	
4, 2, 8, 6, 24, _____, _____	
3, 5, 8, 13, _____, _____	
1, 2, 4, 7, _____, _____	
2, 8, 32, _____, _____	
40, 20, 10, _____, _____	

Jordan is not your typical math kid. Maybe you are not either. He uses drawings or sketches to see the pattern before he tries to solve a sequence.

AND THE DIFFERENCES FORM TRIPLES TOO! WE ARE BOTH RIGHT!

MENU

Do you see what I see? Jordan could draw two more shaded bars to show "second differences". These would be the differences of the differences. And these (6 -2 = 4, 18 -6 = 12) form triples too. Hold on, I am not sure that even Keanu knows this.
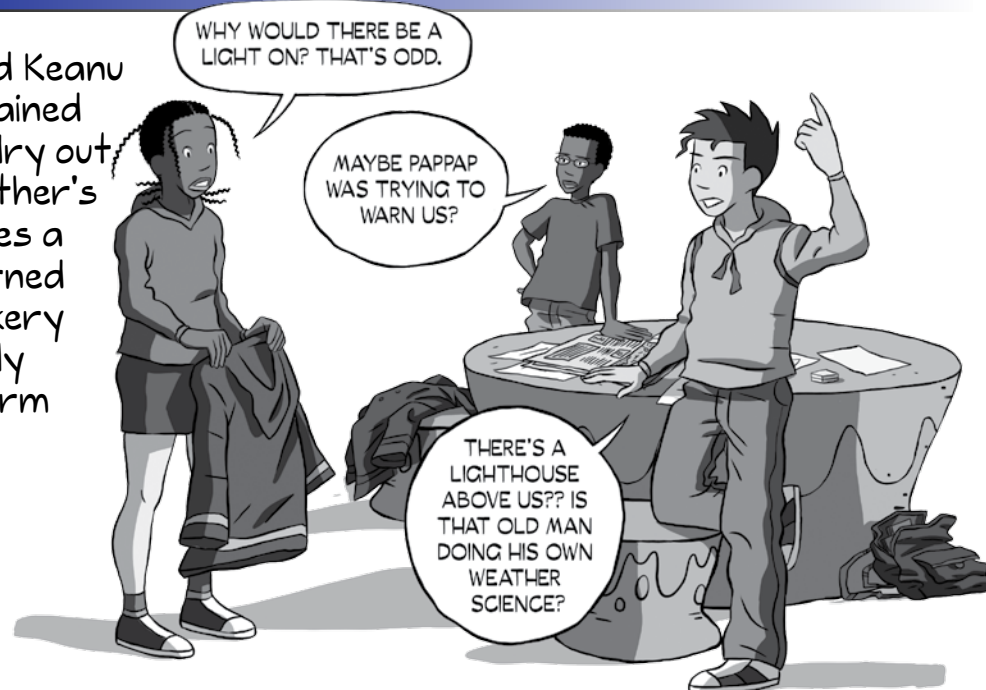
Exercises.

Please solve the sequences below by considering the pattern of differences, or second differences, if needed.

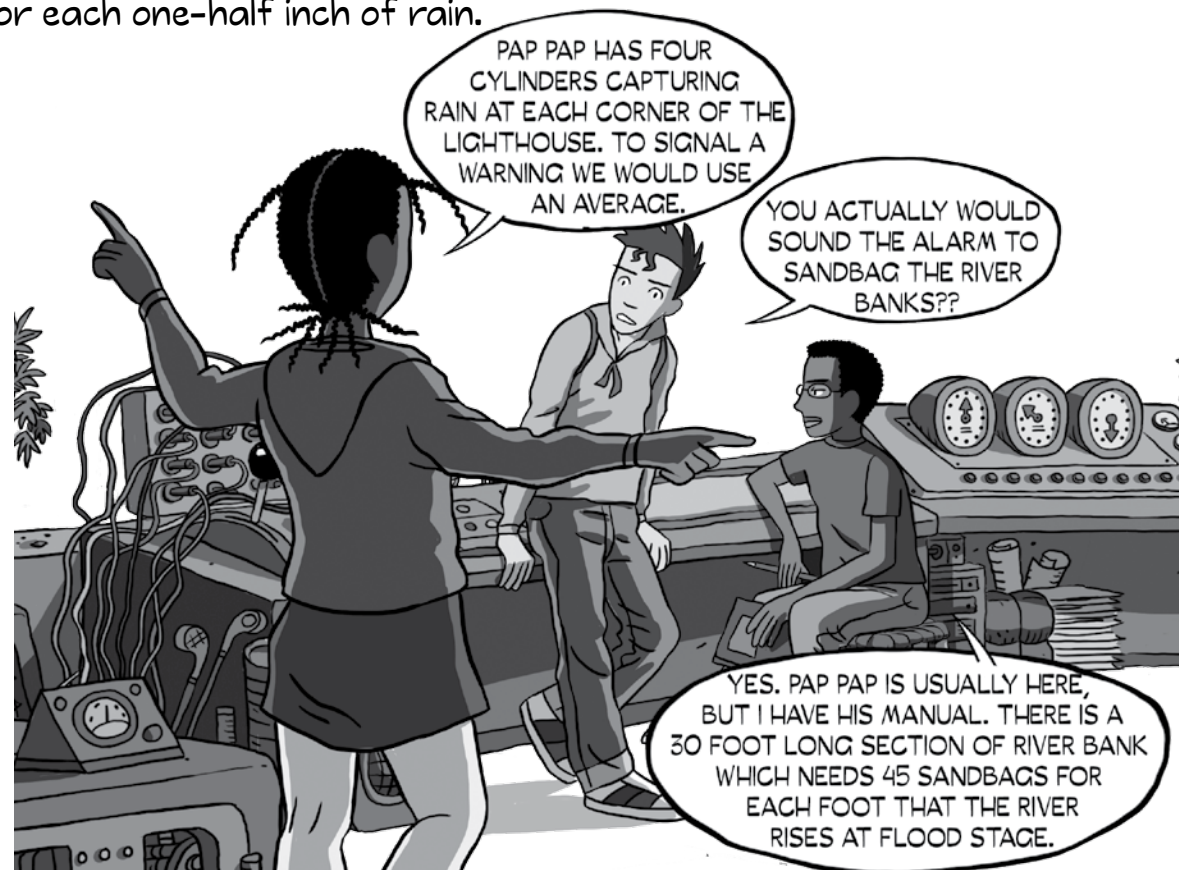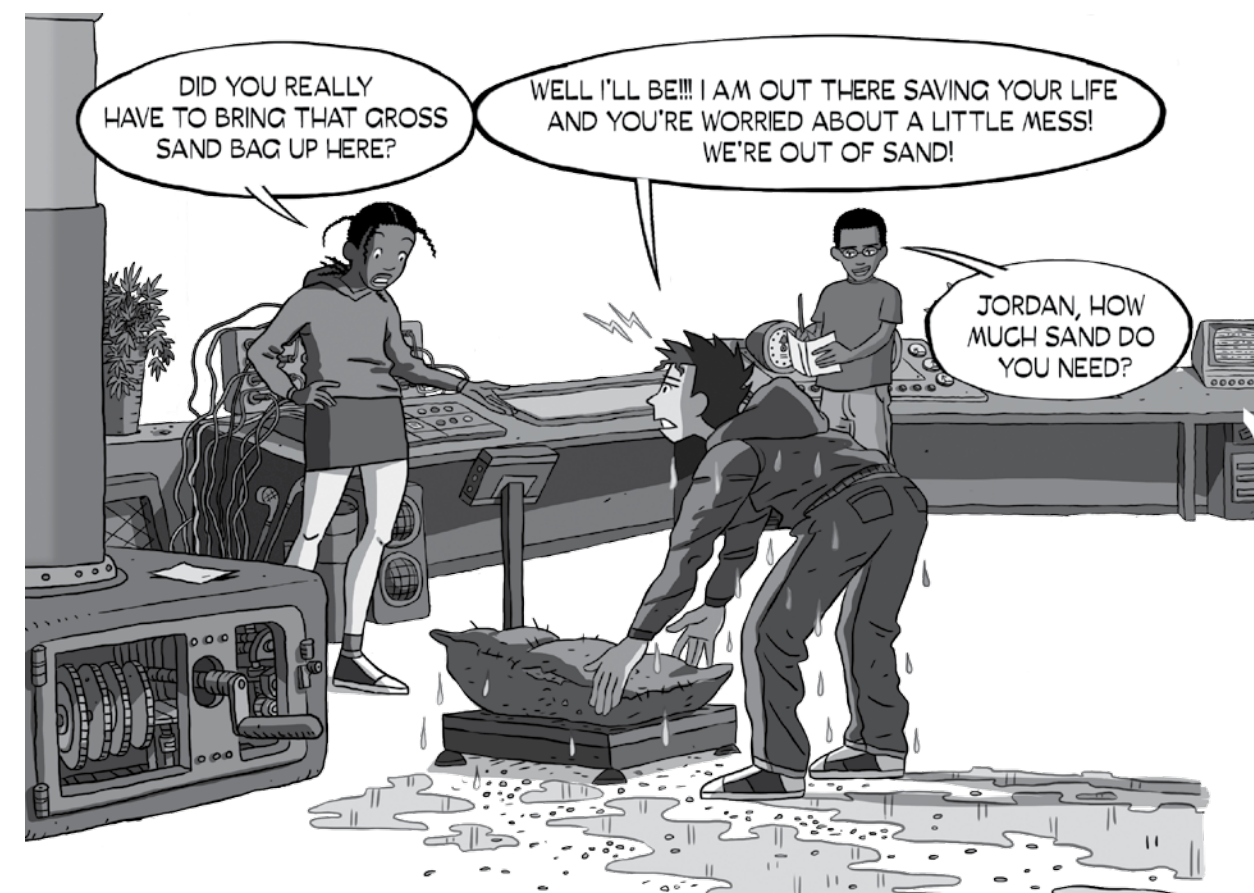| Sequence	Rule
8, 6, _____, _____, 0	
7, _____, _____, 27, 34	
14, _____, _____, 6.5, 4	
18, 30, _____, _____, 66	
30, 24.5, _____, _____, 8	
0, _____, _____, 18, 24	

Lucy, Jordan and Keanu return from a rained out bike trip to dry out in their grandfather's bakery. Lucy sees a warning light turned on above the bakery and that can only mean that a storm is brewing.

WHY WOULD THERE BE A LIGHT ON? THAT'S ODD.

MAYBE PAPPAP WAS TRYING TO WARN US?

THERE'S A LIGHTHOUSE ABOVE US?? IS THAT OLD MAN DOING HIS OWN WEATHER SCIENCE?

Indeed, the skies go from gray to black. The lighthouse is equipped with four rain gauges so that no one gauge full of windblown water is used for a warning. The gauge levels at noon are at 1.25, 1.25, 1.5 and 2 inches. Above this average of 1.5 inches in the gauges, the river will rise two feet for each one-half inch of rain.
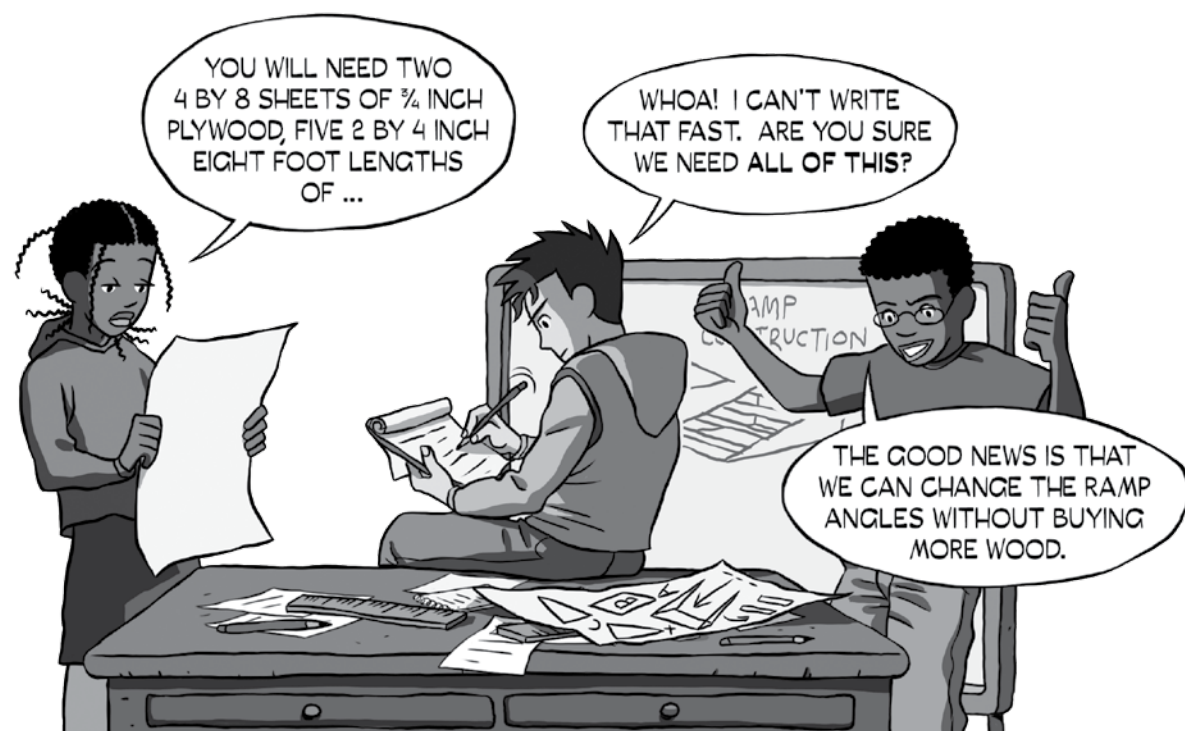
PAP PAP HAS FOUR CYLINDERS CAPTURING RAIN AT EACH CORNER OF THE LIGHTHOUSE. TO SIGNAL A WARNING WE WOULD USE AN AVERAGE.

YOU ACTUALLY WOULD SOUND THE ALARM TO SANDBAG THE RIVER BANKS??

YES. PAP PAP IS USUALLY HERE, BUT I HAVE HIS MANUAL. THERE IS A 30 FOOT LONG SECTION OF RIVER BANK WHICH NEEDS 45 SANDBAGS FOR EACH FOOT THAT THE RIVER RISES AT FLOOD STAGE.

At six feet above normal, the river will be at flood stage and the alarm to begin sandbagging will be sounded. If it's raining at the rate of one inch per hour, how many sandbags will be needed to keep the river in its banks until 4 pm?

I SET THE SIREN TO SOUND IN SHORT BLASTS. LET US DO THE MATH. YOU'RE NEEDED TO BUILD THOSE SAND BAG WALLS AND SAVE US, REMEMBER?

BUT SURELY THE MATH IS MORE IMPORTANT?

DID YOU REALLY HAVE TO BRING THAT GROSS SAND BAG UP HERE?

WELL I'LL BE!!! I AM OUT THERE SAVING YOUR LIFE AND YOU'RE WORRIED ABOUT A LITTLE MESS! WE'RE OUT OF SAND!
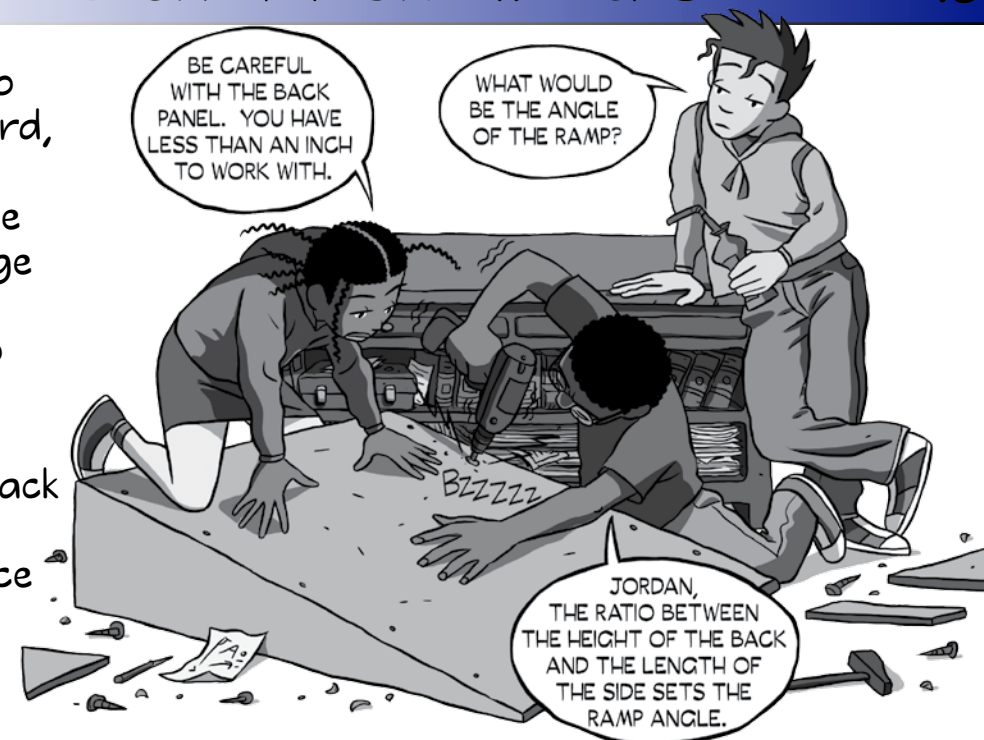
JORDAN, HOW MUCH SAND DO YOU NEED?

How can Jordan answer Keanu's question?

Every skateboard ramp has an angle that determines its slant. It is known that the greater the measure of the angle, the steeper the slant and the more thrilling the ride.

HELLO! HUNGRY? GET YOURSELF SOMETHING OUT OF THE FRIDGE. WHAT DO YOU HAVE?

ARE YOU THINKING THAT WE ARE GOING TO USE **ALL OF THAT?**

YES. I'M GOING TO TEACH YOU **TO SKATEBOARD!** BUT FIRST WE NEED RAMPS.

YOU WILL NEED TWO 4 BY 8 SHEETS OF ¾ INCH PLYWOOD, FIVE 2 BY 4 INCH EIGHT FOOT LENGTHS OF ...

WHOA! I CAN'T WRITE THAT FAST. ARE YOU SURE WE NEED **ALL OF THIS?**

THE GOOD NEWS IS THAT WE CAN CHANGE THE RAMP ANGLES WITHOUT BUYING MORE WOOD.

To build ramps, the kids use a sketch that becomes a scale drawing for each size. Lucy tells an unhappy Jordan what to buy, but Keanu has some good news about building additional ramps.

After a trip to the lumber yard, the kids begin assembling the ramp, or wedge as it's called. There are two triangular side panels, a rectangular back panel and the skating surface which is 3 feet long and four feet wide.

BE CAREFUL WITH THE BACK PANEL. YOU HAVE LESS THAN AN INCH TO WORK WITH.

WHAT WOULD BE THE ANGLE OF THE RAMP?

JORDAN, THE RATIO BETWEEN THE HEIGHT OF THE BACK AND THE LENGTH OF THE SIDE SETS THE RAMP ANGLE.

BZZZZZ

HERE YOU GO, JORDAN, TWO BEAUTIFUL WEDGES A GREEN CIRCLE AND A BLUE SQUARE.

DOES ANYONE HAVE A **STOPWATCH?** I NEED TO TRY OUT MY, EXCUSE ME, OUR WEDGES TO CHECK **THE SPEEDS.**

I HAVE JUST WHAT YOU NEED, **JORDAN.**

Green circle wedges have angles from 9 to 12 degrees; Jordan skated these at speeds from 3.0 to 3.5 meters per second. Blue square wedges have angles from 16 to 21 degrees; Jordan had speeds of 3.5 to 4.0 on these. If Lucy and Keanu were to build black diamond wedges for speeds between 5.0 and 5.5, what angles would they use assuming the same pattern continued?

Keanu's mom was careful in setting neighborhood boundaries for him. When he was six, he was allowed only on his own block, which was bounded by Maple and Elm Streets running east and west, and Fourth and Fifth Avenues running north and south. But he was not permitted to be on any alley.
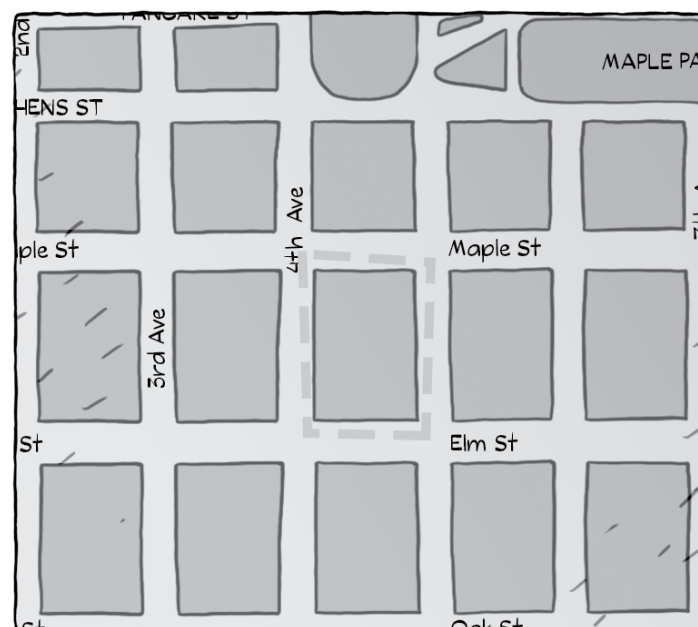
This was an area of one square block. Each year he was permitted more territory. First his mom allowed him to extend between between Third and Sixth Avenues; then he could go between Oak and Elm Streets the next year. By the time he was nine, he also could travel on his own between Second and Seventh Avenues. (Note the boundary in the map to the right)
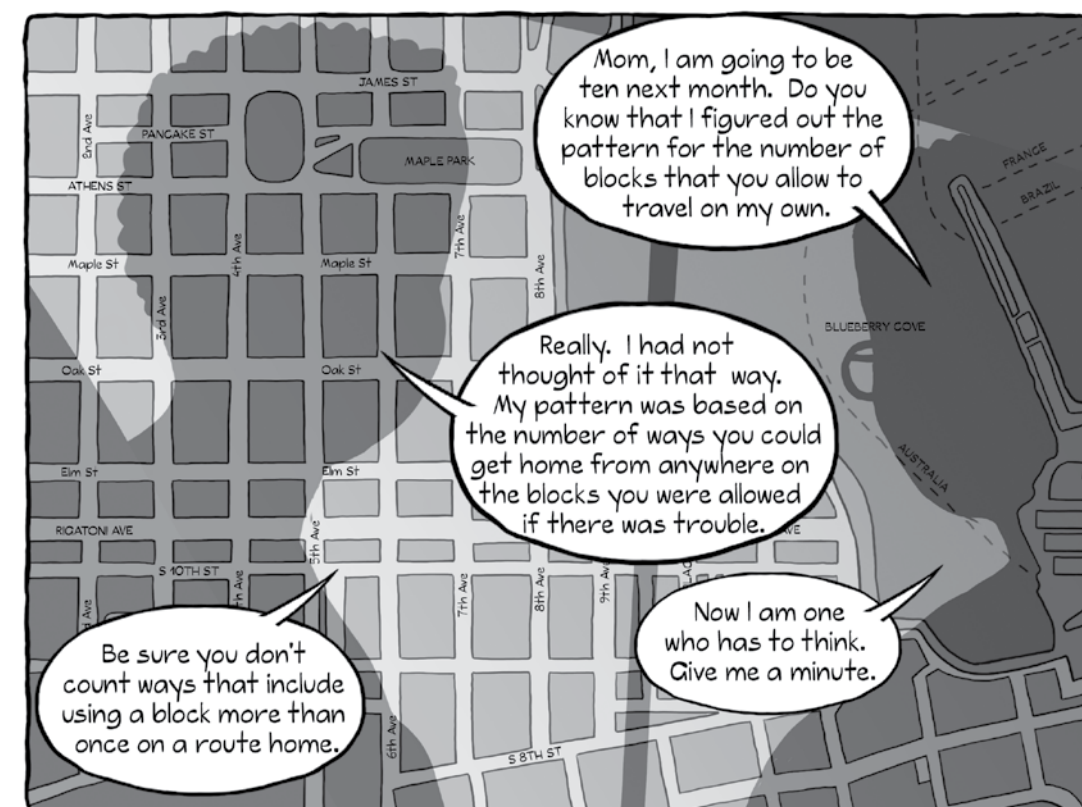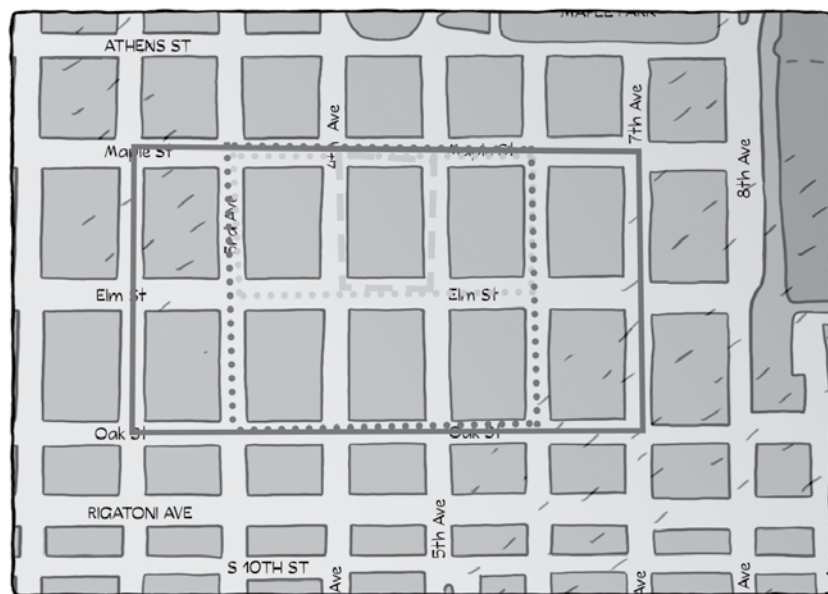
Lucy and Keanu's grandfather, PapPap, owns the bakery where they often meet. One Saturday morning he asks them to run the shop (Jordan too). PapPap is concerned that there are always enough muffins for his customers.

Keanu begins to work on organizing what needs to be done to run the shop. Lucy comes out of the kitchen to fetch her sheet of duties only to smell the wrong aroma.

PapPap wanted Lucy's two batches for the parade of customers that he expected from 7 am to noon. With no muffins for the 7 am crowd, how many should Lucy bake?

It looks like the kids came up with a solution as the bins are bare as Keanu reduces prices.
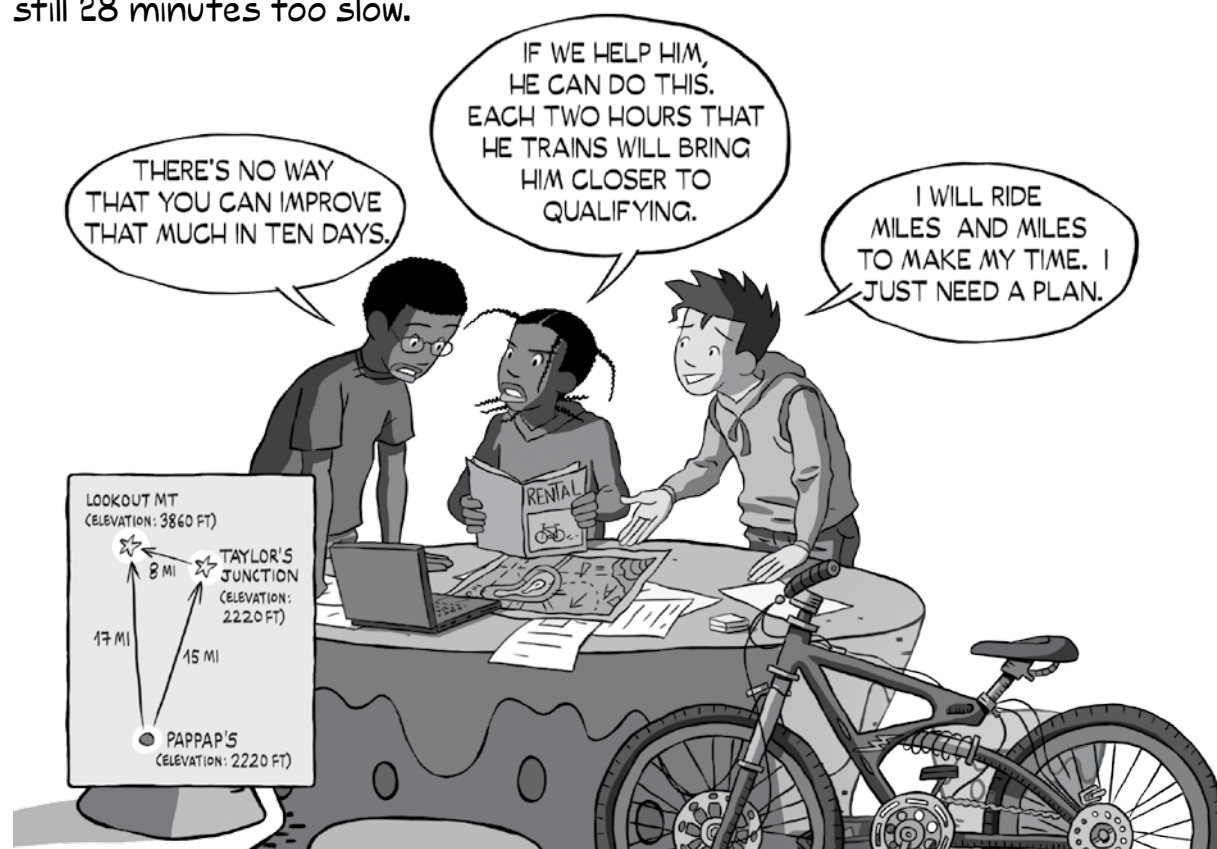
Jordan likes to bike and wants to join a cycling team. To become a team member, he must ride 10 kilometers in 48 minutes. Without any training, he can do this ride in 84 minutes, or 36 minutes slower than is required.

SO YOU WANT TO BE A MEMBER OF THE BLAZERS CYCLE TEAM?

HE'LL BE GREAT. HAVE YOU SEE HIM RIDE? HE'S THE FASTEST KID IN OUR GRADE.

BUT THE TEAM HAS OLDER RIDERS, AND MY TIME FOR 10 KILOMETERS IS 36 MINUTES TOO SLOW. AND I ONLY HAVE TEN DAYS TO QUALIFY.
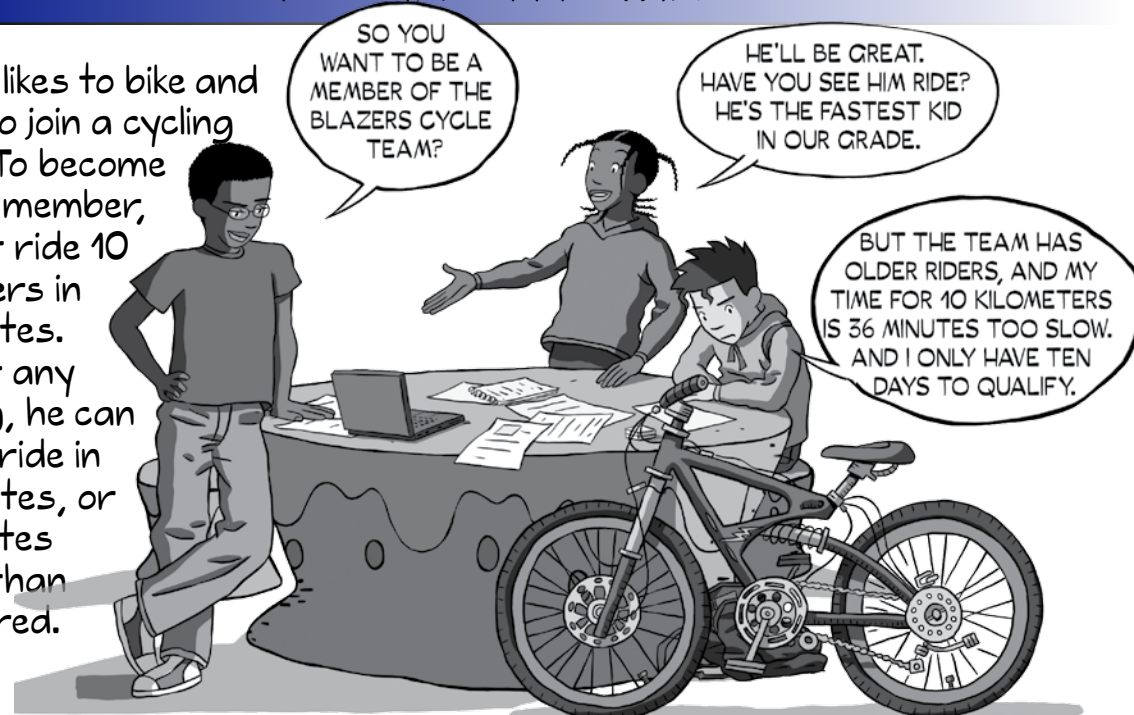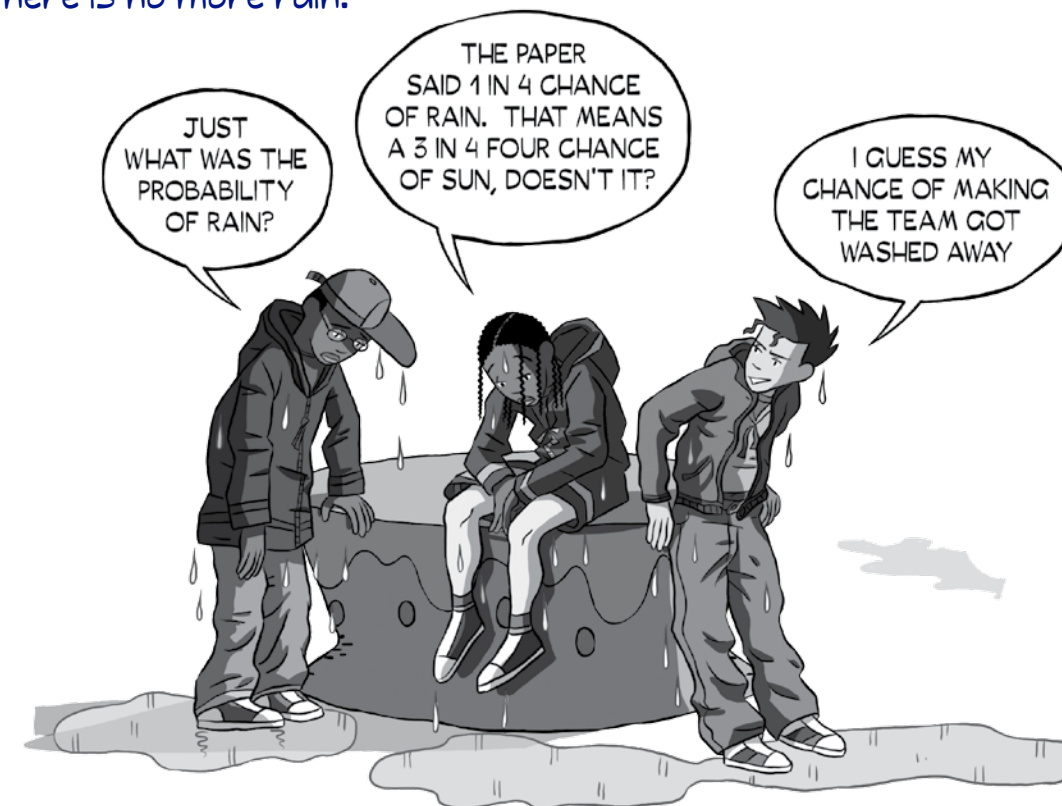
To become a team member, Jordan will need to plan his training and begins at two hours a day on flat routes. But after two days of riding two hours a day, his times at 10 kilometers are disappointing and he is still 28 minutes too slow.

THERE'S NO WAY THAT YOU CAN IMPROVE THAT MUCH IN TEN DAYS.

IF WE HELP HIM, HE CAN DO THIS. EACH TWO HOURS THAT HE TRAINS WILL BRING HIM CLOSER TO QUALIFYING.

I WILL RIDE MILES AND MILES TO MAKE MY TIME. I JUST NEED A PLAN.

LOOKOUT MT (ELEVATION: 3860 FT)
TAYLOR'S JUNCTION (ELEVATION: 2220 FT)
8 MI
17 MI
15 MI
PAPPAP'S (ELEVATION: 2220 FT)

I AM WORKING ON THE PLAN TO USE THE HILLY NORTHWEST ROUTE. HE'LL BEGIN BY BEING FASTER BY 8 MINUTES EACH OF THE FIRST TWO DAYS; THEN IF HE CONTINUES HE'LL BE FASTER BY HALF AS MUCH EACH SECOND DAY.

LET ME SEE. IF WE TAKE THE NORTH ROUTE FIRST TO GET USED TO THE BIKES BEFORE WE TAKE ON THE MOUNTAIN CLIMB, HOW MUCH TIME DOES THAT COST US?

Jordan knows that cycling on hills will make him a faster cyclist than training on flats as he has been doing. To the rescue comes Team Jordan, which is Lucy and Keanu.

DO WE NEED TO CHECK THE WEATHER FORECAST?

Jordan has two days washed out due to rain. And he can only do two days in a row of the hills before needing a day of cycling on flats to recover. Show with a diagram and your math whether he can qualify if there is no more rain.

JUST WHAT WAS THE PROBABILITY OF RAIN?

THE PAPER SAID 1 IN 4 CHANCE OF RAIN. THAT MEANS A 3 IN 4 FOUR CHANGE OF SUN, DOESN'T IT?

I GUESS MY CHANCE OF MAKING THE TEAM GOT WASHED AWAY

Jordan's brother, Sean, is a drummer. When he plays in his attic, Jordan's mother can be heard screaming for him to stop. The kids agree to soundproof the attic in exchange for playing in the band: Lucy as the singer; Jordan on electric guitar; and Keanu on drums.

Mom insists that the band reduce the noise level from 84 decibels to no more than 36 dbs for them to play.
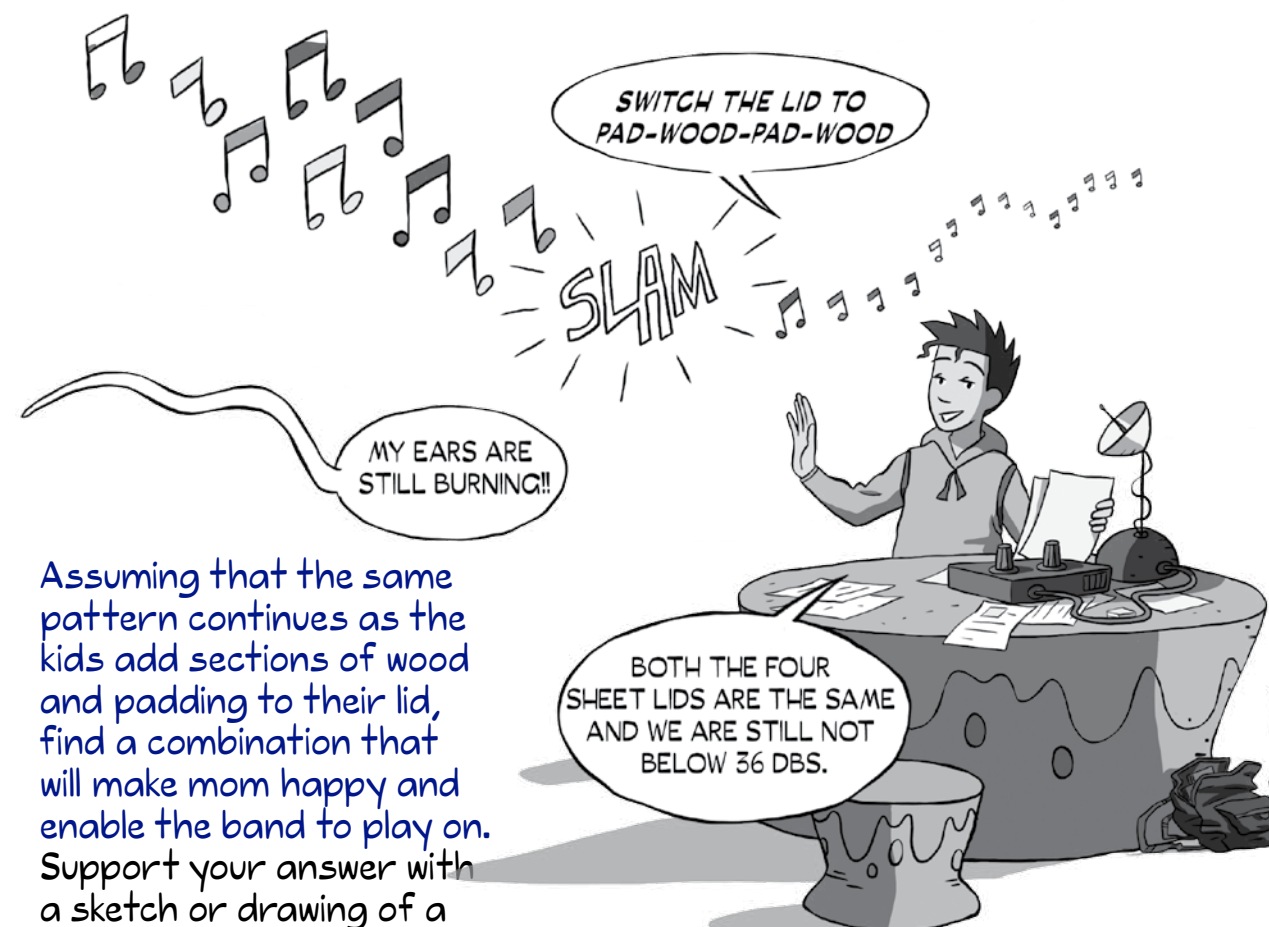
Lucy is very clever and wants to build a sandwich with padding between two sheets of wood to form the lid on the staircase.

OBVIOUSLY WE HAVE TO BUILD A COVER TO THE STAIRWELL. BUT WHAT WILL THAT BE ENOUGH?

WE'LL USE A COMBINATION OF WOOD AND PADDING. THE FIRST WOOD SHEET WILL GIVE US A 24 DECIBEL REDUCTION AND THE PADDING ANOTHER 12 DBS.

BUT YOUR MOM WANTS THE MUSIC TO BE LESS THAN 36 DBS. CAN WE GIVE HER SOME EAR PLUGS?

Keanu knows that each additional piece of wood or padding has less and less effect. And he has found a pattern in the noise reductions for additional sheets to the lid, or the sandwich.

The only way to test Keanu's idea is to play some music in the attic. The kids borrow mom's sound meter to test the noise level with a wood-pad-wood-pad lid. Jordan comes downstairs to talk with his mom and test the noise as Lucy and Keanu play and sing in the attic.

LET'S BUILD A WOOD-PAD-WOOD LID.

LUCY, GREAT IDEA. OR COULD WE GO WITH PAD-WOOD-PAD?

THE SECOND SHEET OF WOOD OR PADDING WOULD ONLY GIVE US ANOTHER 6 DBS REDUCTION. WE NEED TO TRY ANOTHER SHEET.

SWITCH THE LID TO PAD-WOOD-PAD-WOOD

SLAM

MY EARS ARE STILL BURNING!!

BOTH THE FOUR SHEET LIDS ARE THE SAME AND WE ARE STILL NOT BELOW 36 DBS.

Assuming that the same pattern continues as the kids add sections of wood and padding to their lid, find a combination that will make mom happy and enable the band to play on. Support your answer with a sketch or drawing of a sequence of lids.

These notes are offered to you as a help in solving the exercises and problems. Often your solution, or the path you take to solve a sequence, will be different from mine. Pap didn't argue with Keanu because both solutions worked. Please don't argue with me, but you can question me. Sadly, I don't bake brownies.

DO YOU SEE A PATTERN? PAGE 4

#2 Here is the classic Fibonacci sequence.

#4 This is a classic geometric with a rate of 1/2. I think that this may be the sequence that Keanu was viewing on Getting Started, page 2.

#5 Another classic. Actually this is just a portion since the squares will be of the odd numbers beginning with 3.

#6 The rule here is complicated. Be sure to test every term of your answer.

#9 There should be four sequences that were not used in your answers. Please begin your answer here with "Each term.."

DO YOU SEE A PATTERN?   PAGE 5

#1 This is a classic geometric with a rate of 2.

#6 This is just a portion of one of the classics.

#8 There are two maths in the pattern here. You'll need both to write the rule and solve it.

#9   The differences jump dramatically. Think multiplication to find the pattern.

THE KEY IS THE DIFFERENCE   PAGE 6

#4 and #6 You may want to ask another student about decimal numbers if you have only worked with whole numbers up to this point. I also suggest that you draw a number line and mark these points on it so you can see the solution.

THE KEY IS THE DIFFERENCE   PAGE 7

Lucy and Jordan were such a great help in this section that I have little to add. If decimal numbers are a problem, please follow my advice above.

THE KEY IS THE DIFFERENCE   PAGE 8

Both #1 and #2 use two maths in the rule for the sequence. With a little work, you should be successful.

All the rest are classics except #4. For its solution, look at the pattern of differences.

THE KEY IS THE DIFFERENCE   PAGE 9

The first five sequences can be solved rather easily. But stronger students may have to allow more time with the last three.

THE PROBLEMS

Jordan was wrong when he said a few pages ago, "NOW WE HAVE A DECENT PROBLEM." He was speaking about solving a classical arithmetic sequence with two missing terms. Lucy quickly spoke up and told everyone who was listening, hopefully that was you too, how to solve such a sequence. So we included a few in the Exercises, as we left Keanu to figure out if he could use averages to solve them.

The pages you just started (or completed) hopefully had problems that may require you to think. To complete these solutions, I will need to see a drawing, sketch, diagram or table showing that you understand the math. If you are not certain what I mean, please read these notes.

A STORM IS BREWING
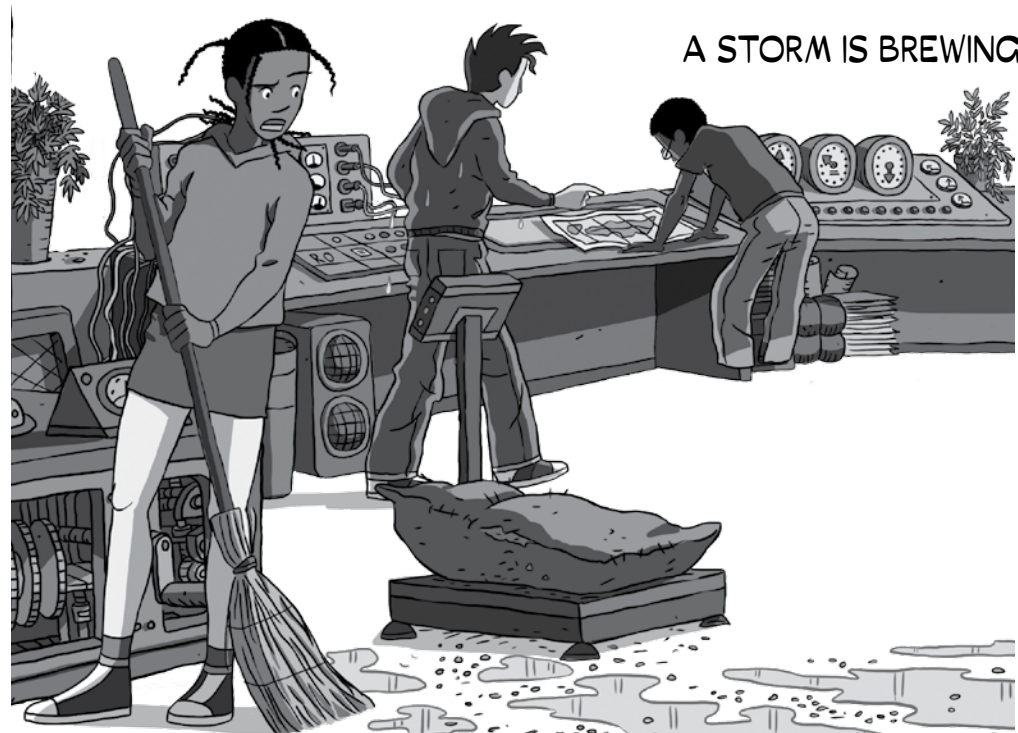
Poor Lucy is left to clean up the mess from Jordan's sand bag. Students have a good deal of math to do here so that they can see the rule between the rain and the flooding of the river. Once they discover this, they need to link the amount of flooding to the number of sandbags.

Drawings are important to the solution. In fact, you can see Jordan and Keanu working off a drawing to solve the problem themselves.

In case you wanted to answer the question by weights, each sandbag weighs 40 pounds.

BLACK DIAMOND WEDGES
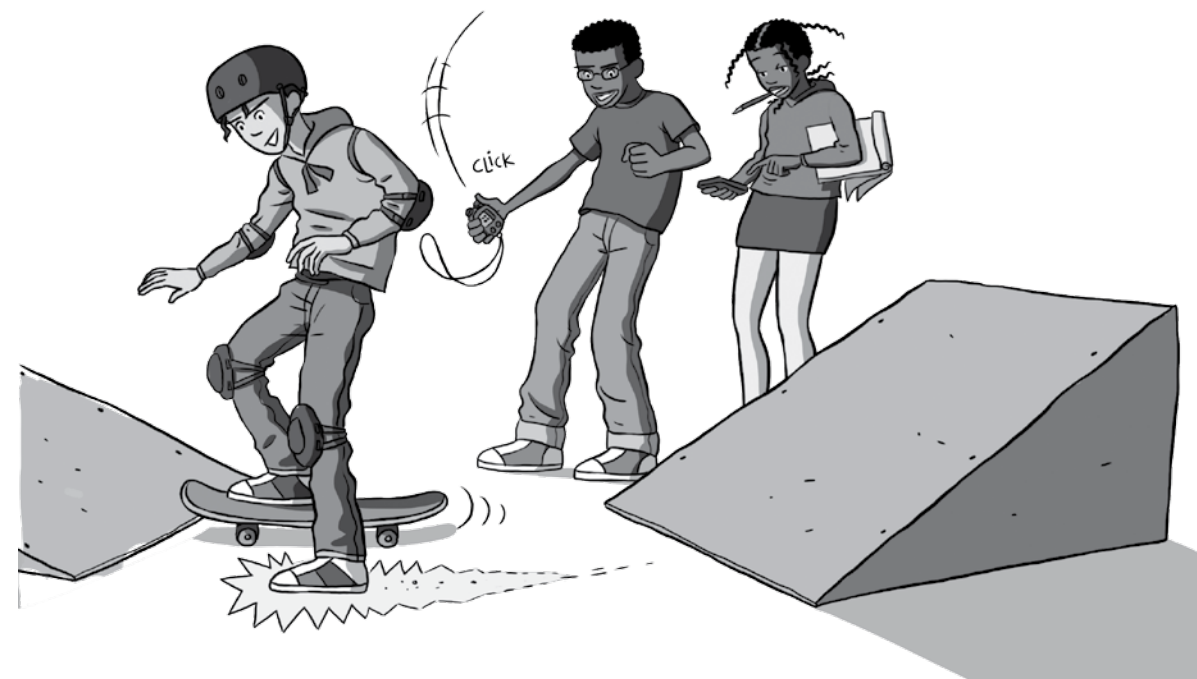
Lucy and Keanu have the right idea to record Jordan's speeds on each of several different wedges. They do this by using a stop watch to get his time. The wedge surfaces are each 3 feet so it's a simple matter to calculate his speed.

Successful students have used a table to record the relationship between the angle of the wedge and the speeds he can skate. Others have sketched the wedges from the ones that are almost flat to the steeper ones and recorded the speed under each. Remember to start with the 9 degree wedge and a speed of 3.0 meters per second.

Do not attempt to actually build the wedges and do the skating unless you are a very accomplished skateboarder.

ONCE AROUND THE BLOCK
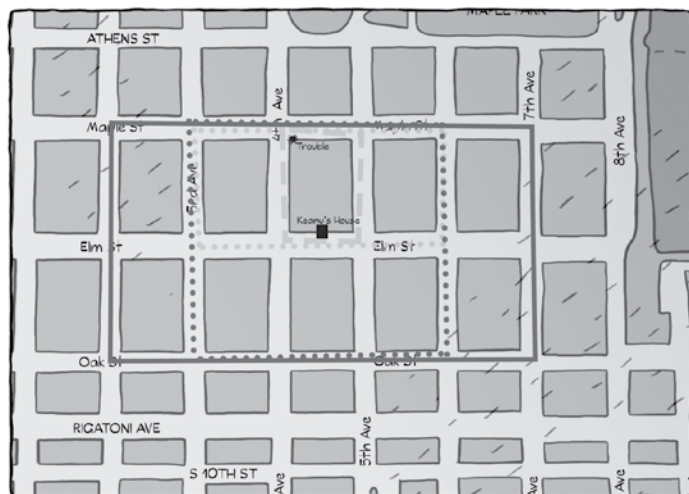
Everyone thinks that this is a relatively simple problem when they consider the sequence of blocks that Keanu has permission to be on each year. But mom introduces the notion of how many ways home from any position in the permitted territory. The first year, Keanu is allowed on a single block and there are two ways home from any trouble. But be careful when he's older, more blocks and now alleys. Good luck.

MUFFINS ANYONE?

Indeed, Lucy is our heroine, especially after she ruined the first batch.

To solve this problem, you will need to show an hour by hour chart matching the number of customers each hour whom Pap thought would eat all the two batches of muffins. Then show the math that you can use to create a new sequence beginning at 8 am. When you find your answer for Lucy, remember that she needs to bake in dozens.

RIDE LIKE THE WIND

Jordan seems discouraged about being able to join the bike team. Can you improve his attitude by showing him the way he can qualify or find that there is no way?

You will need to sort through all of the information and devise a plan that considers increases in speed and better 10K times for training on hills and flats, as well as delays for rain.
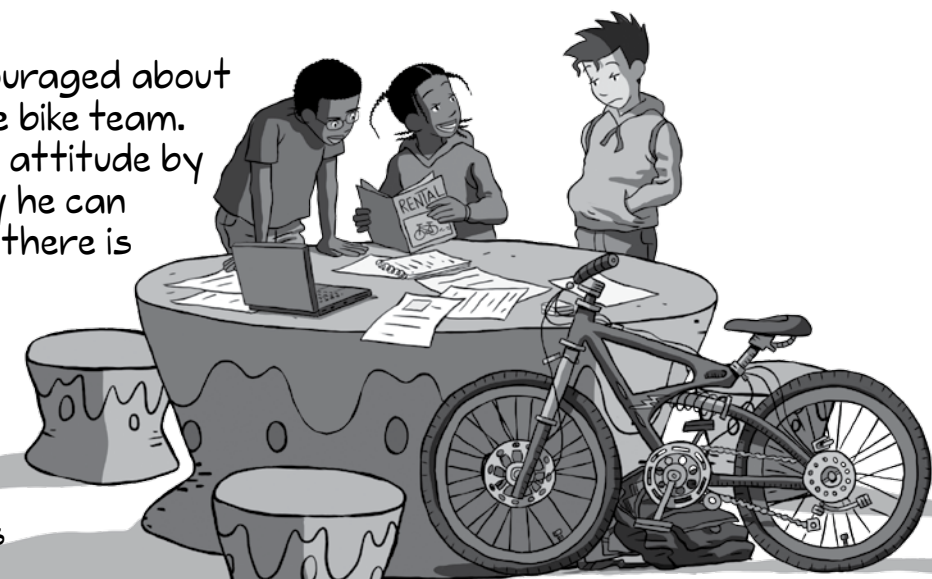
THE BAND PLAYS ON

Here's Lucy pounding a wood-pad-wood lid for her band. Good students have sketched a series of these and marked the amount of noise reduction for each beginning at 24 dbs for a lid with a single piece of wood or pad.

Your solution should have drawings and math to show that mom will let the band play on.